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SIDRAUSKI MODEL WITH ENDOGENOUS LABOR[1]

Abstract

In the paper we analyze the Sidrauski model similar by Walsh (2003) (see for example Sterken, 2005). For analyses it is needed to use Dynamic optimization techniques. The model is extended by endogenous labor. We show that all Walsh's conditions are kept. Even though in steady state analysis in generally the superneutrality of money theory does not hold. Labor supply, is no longer inelastic and influences marginal productivity. To reach the money superneutrality we must assume that preferences of representative consumer are represented by separable utility function. A similar analysis was made by Walsh, the results are the same in general case.

Keywords

Sidrauski Model, Labor, Dynamic Programming, Envelope Theory, Steady State, Supernautrality of Money, Separable Utility Function, Friedman (Chicago) Rule JEL: C02, E13

ACM classification

J.4 SOCIAL AND BEHAVIORAL SCIENCES, *Economics,* I.2.8 Problem Solving, Control Methods, and Search, *Dynamic programming*

JEL classification

C6 – Mathematical Methods and Programming, C61 – Optimization Techniques; Programming Models; Dynamic Analysis

INTRODUCTION

Walsh (2003), using Sidrauski model, showed that in monetary economy there is *superneutrality of money*. In steady state the money growth in the long run does not influence the real variables. In such environment the Friedman (Chicago) rule of zero nominal interest rates for optimal money growth holds. By the same analysis we test the Sidrauski model extended by endogenous labor (labor in the utility function). This analysis needs to be home with the Bellman dynamic programming. Short introduction of the dynamic programming is in the first section. In the second we form and solve the model. In the third section is the steady state analysis. Even though three expressions solving our model are the same as Walsh's, we cannot accept the superneutrality of money theory. Fourth condition, labor supply, puts that labor depends on money growth. We will show that for example if utility function of representative consumer, expressing his welfare, is separable the superneutrality of money holds. Finally we will demonstrate that optimal money growth is given by the Friedman (Chicago) rule.

1. DYNAMIC OPTIMIZATION

„Consider a fish stock which has some natural rate of growth and which is harvested. Too much harvesting could endanger the survival of the fish, too little and profits are forgone. The obvious question is: 'what is the best harvesting rate, i.e., what is the optimal harvesting?'" (Gandolfo, 1997).

This is typical example of the dynamic optimization problem. Generally we can form the discrete problem as:

$$\max J = \sum_{t=0}^{T} \beta^t U(x_t, u_t)$$

$$x_{t+1} - x_t = f(x_t, u_t)$$

$$x_0 = x^0$$

$$x_T = x^T$$

where t denotes time, u is the control variable, x the state variable.

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There are three types of techniques relevant to economics that provide solution of such problem:

- Calculus of Variation
- Theory of Optimal Control
- Dynamic Programming

For discrete problems a dynamic programming technique can be used. Sterken (2004), for example, has precisely explained all three techniques.

DYNAMIC PROGRAMMING

Define the problem using the value function as:

$$V(x_t)=\max\{U(x_t,u_t)+\beta V(x_{t+1})\}$$

Subject to:

$$x_{t+1}=f(x_t,u_t)-x_t$$

By substituting restriction to the value function we get the optimality conditions:

$$\frac{dU}{du_t}+\beta\frac{df}{du_t}\frac{dV(x_{t+1})}{du_t}=0$$

$$\frac{dU}{du_t}=\lambda_t$$

$$x_0=x^0$$

$$\lambda_T\left(x_T-x^T\right)=0$$

The value of slack parameter $\ddot{e}_t$ is given by the envelope theorem (second condition).

2. THE MODEL

FORMULATING MODEL

Now let us develop the model. Suppose the firm has the following production function:

$$Y_t=F(K_{t-1},L_{t-1}) \quad (1)$$

where Y_t is production, K_t is capital stock and L_t is labor in time t. The production function has normal Inada properties (see for example Gandolfo; 1997), $F_K \geq 0$, $F_{KK} \leq 0$, $\lim_{K\to 0} F_K(K)=0$, $\lim_{K\to\infty} F_K(K)=\infty$[2].

Suppose that welfare of representative consumer is determined by utility function:

$$W_t=\sum_{t=0}^{\infty}u(c_t,l_t,m_t) \quad (2)$$

where c_t is per capita consumption, l_t is per capita labor and m_t is per capita holding of money. We assume that u_c, $u_m > 0$ and u_l, u_{cc}, u_{mm}, $u_{ll} < 0$, so the utility function is concave and twice continuously differentiable.

Now let define the budget constraint of consumers as:

$$Y_t+\tau_t N_t+(1-\delta)K_{t-1}+\frac{(1+i_{t-1})B_{t-1}}{P_t}+\frac{M_{t-1}}{P_t}= \\ =C_t+K_t+\frac{M_t}{P_t}+\frac{B_t}{P_t} \quad (3)$$

In time t wealth of consumers consists from nominal wage and capital incomes Y_t; $\tau_t N_t$ units of lump-sum money transfers, where N_t is population size; inherited nominal capital stocks K_{t-1}, inherited real money stocks M_{t-1} indexed by current price index P_t and inherited bonds B_{t-1} evaluated by nominal interest rate $1+i_{t-1}$ and indexed by price index P_t. For these resources consumers buy nominal depreciated capital stocks with depreciation ratio of δ, new capital stocks K_t, new real money stocks M_t/P_t, new real bonds B_t/P_t and fulfill their current consumption demand C_t.

We need to rewrite budget constraint in per capita units. Then consumption of representative agent is $c_t=C_t/N_t$, his capital stocks $k_t=K_t/N_t$, labor $l_t=L_t/N_t$, real money $m_t=M_t/P_tN_t$ and bonds $b_t=B_t/P_tN_t$. We assume that the production function is linear homogenous of degree one, so $F(\lambda K,\lambda L)=\lambda Y$. We know that

$$\frac{K_{t-1}}{N_t}=\frac{K_{t-1}}{N_{t-1}}\frac{N_{t-1}}{N_t}=\frac{K_{t-1}}{N_{t-1}(1+n)}=\frac{k_{t-1}}{1+n}$$

where n is population growth rate:

$$1+n=\frac{N_t}{N_{t-1}}$$

Similarly $L_{t-1}/N_t=l_{t-1}/(1+n)$ and so the production function per capita is:

$$y_t=f\left(\frac{k_{t-1}}{1+n},\frac{l_{t-1}}{1+n}\right) \quad (4)$$

We know:

$$\frac{(1+i_{t-1})B_{t-1}}{P_tN_t}=\frac{(1+i_{t-1})B_{t-1}}{P_{t-1}N_{t-1}}\frac{P_{t-1}N_{t-1}}{P_tN_t}=\frac{(1+i_{t-1})b_{t-1}}{(1+\pi_t)(1+n)},$$

where π_t is current inflation rate.

[2] By symbol F_K, for example, we denote first partial derivation of function F according to variable K, F_{KK} is second partial derivation of function F according to variable K. In the next text we will use similar denotation of partial derivations.

So by dividing (3) by the population size Nt we gain the wealth of representative consumer ω_t:

$$\omega_t = f\left(\frac{k_t-1}{1+n}, \frac{l_t-1}{1+n}\right) + \tau_t + \frac{1-\delta}{1+n}k_t-1 + \frac{(1+i_t-1)b_t-1+m_t-1}{(1+\pi_t)(1+n)} = \\ = c_t + k_t + m_t + b_t \qquad (5)$$

Representative consumer maximizes his utility function (2) with respect to his budget constraint (5). Walsh solves such problem using dynamic programming technique (see section 1).

SOLVING THE MODEL

Define the value function as:

$$V(\omega_t) = \max u(c_t, l_t, m_t) + \beta V(\omega_{t+1}) \qquad (6)$$

which we maximize over c_t, l_t, k_t, m_t, subject to next period resources restriction:

$$\omega_{t+1} = f\left(\frac{k_t}{1+n}, \frac{l_t}{1+n}\right) + \tau_{t+1} + \\ + \frac{1-\delta}{1+n}k_t + \frac{(1+i_t)b_t + m_t}{(1+\pi_{t+1})(1+n)} \qquad (7)$$

We express k_t as $\omega_t - c_t - m_t - b_t$ and rewrite model into:

$$\max u(c_t, l_t, m_t) + \\ +\beta V\left[\begin{array}{l} f\left(\frac{\omega_t - c_t - m_t - b_t}{1+n}, \frac{l_t}{1+n}\right) + \tau_{t+1} + \\ + \frac{1-\delta}{1+n}(\omega_t - c_t - m_t - b_t) + \frac{(1+i_t)b_t + m_t}{(1+\pi_{t+1})(1+n)} \end{array}\right] \qquad (8)$$

so we can now maximize an unconstrained problem over c_t, l_t, m_t and b_t. The first order necessary conditions are:

$$u_c(c_t, l_t, m_t) - \frac{\beta}{1+n}\left[f_k(k_t, l_t) + 1 - \delta\right]V_\omega(\omega_{t+1}) = 0 \qquad (9)$$

$$u_l(c_t, l_t, m_t) + \frac{\beta}{1+n} f_l(k_t, l_t)V_\omega(\omega_{t+1}) = 0 \qquad (10)$$

$$u_m(c_t, l_t, m_t) - \frac{\beta}{1+n}\left[f_k(k_t, l_t) + 1 - \delta\right]V_\omega(\omega_{t+1}) + \frac{\beta V_\omega(\omega_{t+1})}{(1+\pi_{t+1})(1+n)} = 0 \qquad (11)$$

$$\frac{1+i_t}{(1+\pi_{t+1})(1+n)} - \frac{f_k(k_t, l_t) + 1 - \delta}{1+n} = 0 \qquad (12)$$

THE ENVELOPE THEOREM

To gain the envelope condition write the solution to all variables as functions of the state variables: $c_t = c(\omega_t)$, $l_t = l(\omega_t)$, $m_t = m(\omega_t)$, $b_t = b(\omega_t)$ and $\omega_{t+1} = \omega(\omega_t)$. Then we can (8) rewrite as:

$$V(\omega_t) = \max u\left[c(\omega_t), l(\omega_t), m(\omega_t)\right] + \beta V\left\{\begin{array}{l} f\left[\frac{\omega_t - c(\omega_t) - m(\omega_t) - b(\omega_t)}{1+n}, \frac{l(\omega_t)}{1+n}\right] + \\ + \frac{1-\delta}{1+n}\left[\omega_t - c(\omega_t) - m(\omega_t) - b(\omega_t)\right] + \\ + \frac{(1+i_t)b(\omega_t) + m(\omega_t)}{(1+\pi_{t+1})(1+n)} \end{array}\right\} \qquad (13)$$

Partial derivation of $V(\omega_t)$ with respect to ω is:

$$V_\omega(\omega_t) = u_c(c_t, l_t, m_t)c_\omega(\omega_t) + \frac{\beta}{1+n}\left[f_k(k_t, l_t) + 1 - \delta\right]V_\omega(\omega_{t+1})\left[1 - c_\omega(\omega_t)\right] + \\ + u_l(c_t, l_t, m_t)l_\omega(\omega_t) + \frac{\beta}{1+n}\left[f_l(k_t, l_t)\right]V_\omega(\omega_{t+1})l_\omega(\omega_t) + \\ + \left\{u_c(c_t, l_t, m_t) - \frac{\beta}{1+n}\left[f_k(k_t, l_t) + 1 - \delta\right]V_\omega(\omega_{t+1}) + \frac{\beta V_\omega(\omega_{t+1})}{(1+\pi_{t+1})(1+n)}\right\}m_\omega(\omega_t) + \\ + \left[\frac{1+i_t}{(1+\pi_{t+1})(1+n)} - \frac{f_k(k_t, l_t) + 1 - \delta}{1+n}\right]b_\omega(\omega_t) \qquad (14)$$

According to (10), the term in the second line of (14) vanish; according to (11), the term in the third line of (14) vanish; according to (12), the term in the third line of (14) vanish and according to (9) the first line in (14) reduces to

$$V_\omega(\omega_t)=\frac{\beta}{1+n}\left[f_k(k_t,l_t)+1-\delta\right]V_\omega(\omega_{t+1}) \quad (15)$$

By substituting (9) to (15) we get the envelope condition given by:

$$\lambda_t=V_\omega(\omega_t)=u_c(c_t,l_t,m_t) \quad (16)$$

The transversality conditions are (see section 1):

$$\lim_{t\to\infty}\beta^t\lambda_t k_t=0 \quad (17)$$

$$\lim_{t\to\infty}\beta^t\lambda_t b_t=0 \quad (18)$$

$$\lim_{t\to\infty}\beta^t\lambda_t m_t=0 \quad (19)$$

THE SOLUTION

Solution of the problem is given by conditions (9)-(12) and (16)-(19) and is consistent with Walsh's solution of the problem with exogenous labor.

Equation (9) gives the condition that the marginal utility of holding additional capital must equal to the marginal utility of consumption. Equation (10) gives that the marginal utility of employing one additional hour of labor must equal to the negative value of the marginal utility of consumption. Equation (11) gives the condition for the marginal return on holding money. Equation (12) gives the condition for holding bonds.

Define the real return on capital to be: $1+r_t=f_k(k_t,l_t)+1-\delta$. Equation (10) than shows that

$$1+r_t=\frac{1+i_t}{1+\pi_{t+1}} \quad (20)$$

Nominal interest rate equals to the real interest rates plus future inflation. This result is consistent with the neoclassical conception of the rational expectations.

Let us combine the envelope theorem (15) and (9) to get production and services market equilibrium:

$$\frac{\beta u_c(c_{t+1},l_{t+1},m_{t+1})}{u_c(c_t,l_t,m_t)}=\frac{1+n}{1+r_t} \quad (21)$$

If we combine envelope theory (15), (9) and (11) we will get:

$$u_c(c_t,l_t,m_t)=u_m(c_t,l_t,m_t)+ +\frac{\beta u_c(c_{t+1},l_{t+1},m_{t+1})}{(1+\pi_{t+1})(1+n)} \quad (22)$$

Now let us substitute (21) to (22):

$$\frac{u_m(c_t,l_t,m_t)}{u_c(c_t,l_t,m_t)}=1-\frac{1}{(1+\pi_{t+1})(1+r_t)}$$

Finally substitute (20) to yield for the money demand:

$$\frac{u_m(c_t,l_t,m_t)}{u_c(c_t,l_t,m_t)}=\frac{i_t}{(1+i_t)} \quad (23)$$

Derivation of (23) is the Walsh's one.

Define the real return on labor to be $w_t=f_l(k_t,l_t)$ and substitute it with the envelope theory (15) and with (21) to (10) and we have labor supply:

$$-\frac{u_l(c_t,l_t,m_t)}{u_c(c_t,l_t,m_t)}=\frac{w_t}{(1+r_t)} \quad (24)$$

Equilibrium of economy is given by budget constraint of representative consumer (5), consumption demand on product and services market (21), money demand (23) and labor supply (24).

STEADY STATE

Walsh analyzed steady state solution. Solution of this model without labor supply (24) does not differ from Walsh's solution and so the steady state analysis is same. Steady state is defined as: $V(\omega_t)=V(\omega_{t+1})=V(\omega^{ss})$ and $n=0$. Define the growth of money by:

$$\theta=\frac{M_t}{M_{t-1}}-1$$

In order to keep m^{ss} constant we need $\pi^{ss}=\theta$.

Now according to definition of $\tau_t N_t$, per capita lump-sum money transfers:

$$\tau_t=\frac{M_t-M_{t-1}}{P_t N_t}=\frac{\theta_t}{(1+\pi_t)(1+n)}m_{t-1}$$

Let us rewrite the representative consumer budget constraint (5) in the steady state. In the steady state we assume that bonds of representative consumer in the steady state are zero $b^{ss}=0$:

$$f\left(k^{ss}, l^{ss}\right)+\frac{\theta}{(1+\theta)} m^{ss}+(1-\delta) k^{ss}+ \\ +\frac{m^{ss}}{(1+\theta)}=c^{ss}+k^{ss}+m^{ss} \quad (25)$$

Steady state consumption is then:

$$c^{ss}=f\left(k^{ss}, l^{ss}\right)-\delta k^{ss} \quad (26)$$

Expression (21) puts steady state condition for real interest rate in products and services market:

$$1+r^{ss}=\frac{1}{\beta} \quad (27)$$

Steady state inflation rate:

$$\pi^{ss}=\theta^{ss} \quad (28)$$

What is unique labor equilibrium in the steady state? By combining (24) and (27) the steady state real wage is given by:

$$w^{ss}=-\frac{u_l\left(c^{ss}, l^{ss}, m^{ss}\right)}{\beta u_c\left(c^{ss}, l^{ss}, m^{ss}\right)} \quad (29)$$

Even if expressions (26)-(28) are Walsh's, in generally we cannot accept hypothesis of *money superneutrality*, in which the growth rate of money does not affect the steady state values of the real variables. This is just because of endogenous labor. Labor supply, given by (29) is no longer inelastic. According to (29) steady state money growth can influence labor, consumption as well as real wage and these influence other variables.

EXAMPLE

Assume separable utility function in the sense:

$$u(c_t, l_t, m_t)=v[o(c_t, l_t), p(m_t)] \quad (30)$$

We can then labor supply in the steady state (29) rewrite as:

$$w^{ss}=-\frac{o_l\left(c^{ss}, l^{ss}\right)}{\beta o_c\left(c^{ss}, l^{ss}\right)} \quad (31)$$

We can see that labor supply is not influenced by money in example. Superneutrality of money theory keeps when utility function is separable in the sence of (31).

By this assumption we finally can accept the Lucas's (2000) welfare costs of inflation: „In a monetary economy, it is in everyone's interest to try get someone else to hold non-interest-bearing cash and reserves".

Problem is to find the optimal rate of money growth $\check{c}$ that maximizes state utility:

$$\max_{\theta} u=v\left[o\left(c^{ss}, l^{ss}\right), p\left(m^{ss}\right)\right] \quad (32)$$

with respect to (26) and (31). Since the two conditions are money independent we have first order condition:

$$u_m \frac{\partial m^{ss}}{\partial \theta^{ss}}=0 \quad (33)$$

Marginal utility from money should be zero and according to (23) this holds if nominal interest rate equals to zero, $i^{ss}=0$. This rule is called Friedman, or Chicago rule. The steady state inflation rate must then according to (20) be equal to the negative value of real interest rate.

$$1+\pi^{ss}=\beta=1+\theta^{ss} \quad (34)$$

If we make assumption that utility function is separable, the Walsh's analysis is the same in the both Sidrauski models – with exogenous and endogenous labor.

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Autobiographical note

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