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FUTURE VALUE OF AN ANNUITY – CERTAIN AS A SOLUTION OF A DIFFERENCE EQUATION

Abstract
The aim of this note is to show how to use the methods of difference calculus in mathematics of finance. We calculate an accumulated amount of some special immediate annuities by solving the special type of non-homogenous linear difference equations of the first order.

Keywords
Accumulated amount, immediate annuity, non-homogenous linear difference equation.

ACM classification
G.1 NUMERICAL ANALYSIS, G.1.8 Partial Differential Equations

JEL classification
C6 – Mathematical Methods and Programming, C63 – Computational Techniques

In this article we will deal with annuities for which the amounts of the payments create an increasing or decreasing sequence. For some annuities in which payments are not all of an equal amount it is simple matter to find the present (or accumulated) value. But there exist annuities for which it is not so easy. We will show procedure or a method how to solve this problem first for easy and well known annuities (amounts of the payments create an arithmetic sequence) and then for annuities which are more complicated.

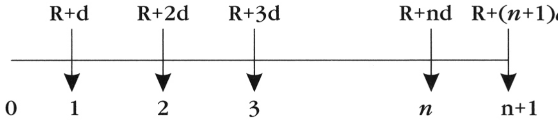
We will solve this problem using methods of difference calculus. So we calculate an accumulated amount of some special immediate annuities by solving the special type of non-homogenous linear difference equations of the first order.

Consider a series of n payments, to be made in arrears at time intervals of one year. Let the amounts of these successive payments form an arithmetic sequence, where the value of the j-th payment is

$$X_j = R + jd,$$

$$\text{for } j = 1, 2, \ldots, n.$$

Let the first payment be $R+d$, difference of the arithmetic sequence is d. Such a sequence of payments is called an annuity-certain or an immediate annuity and is illustrated in the time diagram bellow.



The value of the series of payments at the time the last payment is made is in mathematics of finance called an accumulated amount (sometimes also called future value) of an immediate annuity and is denoted by $s_{\overline{n}|}$. We will try to calculate value of $s_{\overline{n}|}$ (in what follows we denote $s_{\overline{n}|}$ only as s_n) using the methods of difference calculus by solving the special type of difference equation.

First we need recurrence formula for determining a sequence of future values $\{s_n\}_{n=1}^{\infty}$ of this annuity. Denoting yearly interest rate i, then from the definition of this annuity we obtain the following formula

$$s_{n+1} = s_n(1+i) + R + (n+1)d.$$

For simplicity we will use notation $1+i = r$.

But preceding recurrence formula is also a linear difference equation of the first order for the unknown function s_n of the variable n. We rearrange it in the form

$$s_{n+1} - s_n r = R + (n+1)d. \tag{1}$$

and obtain a basic form of the first order linear difference equation with the constant coefficients and with the special form of the right side.

We know [5] that the general solution of the difference equation with (nonzero) right side (or homogenous difference equation) is a sum of the general solution (we denote it y_n) of the equation without the right side and a particular solution (we denote it z_n) of the equation with the right side (or non-homogenous difference equation).

We begin to determine the general solution y_n of the following equation without the right side $S_{n+1} - S_n r = 0$.

Auxiliary equation [4] of this difference equation is $\lambda - r = 0$ and its solution is $\lambda = r$. Thus the general solution y_n of the equation without the right side [5] is

$$y_n = cr^n. \tag{2}$$

We now restrict our attention to find particular solution of (1) for special type of the right side q_n. We first rearrange the right side in the following form

$$q_n = R+(n+1)d = dn+(R+d).$$

Again we know [5] that in the case when on the right side is first order polynomial of the unknown n and number 1 is not the root of auxiliary equation, then the particular solution has a form

$$z_n = An+B,$$

where A, B are unknown constants, which we can find by substituting the particular solution z_n and $z_{n+1} = A(n+1)+B$ in the equation (1)? We then obtain

$$A(n+1)+B-(An+B)r = dn+(R+d).$$

We regroup terms on the left side to obtain coefficients for each power of n. Thus

$$A(1-r)n+A-B(r-1) = dn+(R+d).$$

Then, equating coefficients at the same power of n, we get a system of two equations with two unknowns A, B

$$A(r-1) = -d$$
$$A-B(r-1) = R+d.$$

Solution is

$$A = \frac{-d}{r-1}, \quad B = \frac{-d}{(r-1)^2} - \frac{R+d}{r-1}.$$

Then the particular solution z_n of (1) is

$$z_n = -\frac{d}{r-1}n - \frac{d}{(r-1)^2} - \frac{R+d}{r-1}$$

and the required general solution of (1) is sum of y_n and z_n, i. e.

$$S_n = y_n + z_n =$$
$$= Cr^n - \frac{d}{r-1}n - \frac{d}{(r-1)^2} - \frac{R+d}{r-1} \cdot \tag{3}$$

But we need to find the particular solution which fulfils the additional following conditions: the first payment is $R+d$, that means, we determine parameter C in (3), so that $S_1 = R+d$. Then

$$R+d = Cr^1 - \frac{d}{r-1}1 - \frac{d}{(r-1)^2} - \frac{R+d}{r-1}$$

If we express unknown parameter C from the preceding equation, we obtain

$$C = \frac{R+d}{r} + \frac{d}{r(r-1)} + \frac{d}{r(r-1)^2} + \frac{R+d}{r(r-1)} =$$
$$= \frac{(R+d)(r-1)^2 + (R+d)(r-1) + d + d(r-1)}{r(r-1)^2}$$

and after regrouping some terms

$$C = \frac{R+d}{r-1} + \frac{d}{(r-1)^2}.$$

Hence we substitute C in (3) and we have the required particular solution of our equation which fulfils additional condition and is also a future value of an immediate annuity, terms of which increase as the terms of an arithmetic sequence with the first amount $R + d$ and difference d.

$$S_n = \left(\frac{R+d}{r-1} + \frac{d}{(r-1)^2}\right)r^n - \frac{d}{r-1}n - \frac{d}{(r-1)^2} - \frac{R+d}{r-1}. \tag{4}$$

In expression (4) we first group terms containing R and from other terms we take out the factor $\frac{d}{R-1}$. We obtain

$$S_n = \frac{R}{r-1}r^n - \frac{R}{r-1} + \frac{d}{(r-1)}\left(r^n + \frac{1}{r-1}r^n - 1 - n - \frac{1}{r-1}\right)$$

Then we rewrite this result using notions and notations obviously used in mathematics of finance and hence

$$S_n = \frac{R}{r-1}(r^n-1) + \frac{d}{(r-1)}\left(r^n\left(1+\frac{1}{r-1}\right) - \left(1+\frac{1}{r-1}\right) - n\right).$$

Because $1+\frac{1}{r-1} = \frac{r}{r-1}$, preceding formula may be expressed in the in form

$$S_n = R\frac{r^n-1}{r-1} + d\frac{r\frac{r^n-1}{r-1} - n}{r-1},$$

and after substituting $r = 1+i$, we obtain formula well known from the mathematics of finance [2]

$$S_n = R\frac{(1+i)^n - 1}{i} + d\frac{(1+i)\frac{(1+i)^n - 1}{i} - n}{i}.$$

Which we can rewrite in the form

$$S_n = Rs_n + d\frac{\ddot{s}_n - n}{i},$$

where s_n and $\ddot{s}_n$ are notations for the accumulated amount or future value of a unit immediate annuity and a unit annuity-due respectively.

This procedure we may use also for evaluation of the future value of an annuity, the terms of which change as the terms of a geometric sequence. But for "arithmetic" and "geometric" annuity we have also easier method using standard formulas for n-th partial sum of terms of arithmetic or geometric sequence. Importance has our method especially for variable annuities, for which the partial sums create a sequence, which is defined recurrently and not by determining general expression for n-th element of the sequence.

In the following example we will deduce formula for accumulated amount of some special variable annuity.

Example of such an annuity may be as follows. Suppose we have an annuity $\{X_n\}_{n=1}^{\infty}$, terms of which are recursive determined by the formula

1. $X_1 = 1$
2. $\forall n \in N: X_{n+1} = q \cdot X_n + d.$

If $d = 0$, then the sequence is geometric, if $q = 1$, then it is arithmetic. So let $q \neq 0, 1$ and d is nonzero. Then this sequence of annuity amounts is neither arithmetic nor geometric. Using the preceding procedure we can find general expression for the n-th element of the annuity and then again with the use of the same procedure finding also a formula for the accumulated amount. We demonstrate this procedure on the following example.

Let $q = 2$, $d = 1$ and $X_1 = 1$. Then the formula for the recurrence relationship of the members of the annuity is $X_{n+1} = 2 \cdot X_n + 1$.

First we determine the general expression for the n-th element of the annuity. The basic form of difference equation is $X_{n+1} - 2 \cdot X_n = 1$ and the corresponding homogenous equation is $X_{n+1} - 2 \cdot X_n = 0$. Auxiliary equation is $\lambda - 2 = 0$ and has the solution $\lambda = 2$ and that implies the general solution y_n of the homogenous equation $y_n = c2^n$.

We can easy see that the particular solution is $z_n = -1$. Then the required general solution of the non-homogenous equation and also the general formula for the n-th element of the annuity is the sum of y_n and z_n, i. e

$$X_n = c \cdot 2^n - 1.$$

Finally we determine the particular solution which fulfils the additional following condition: the first payment has value 1 and we obtain $X_n = 2^n - 1$.

In the second step we will determine a general expression for the n-th element of the sequence of the n-th partial sum of terms of our annuity that means that we will find formula for accumulated amount S_n for this special type of immediate annuity. In this case recurrence formula for S_n is $S_{n+1} - S_n r = X_{n+1}$ and using preceding formula for the n-th element of the annuity we obtain

$$S_{n+1} - S_n r = 2^{n+1} - 1. \tag{5}$$

Again the corresponding homogenous equation is $S_{n+1} - r \cdot S_n = 0$, the auxiliary equation is $\lambda - r = 0$ and the general solution y_n of the homogenous equation is $y_n = cr^n$.

We now want to find the particular solution of (5) for special type of the right side. But the right side q_n is sum of two terms $q_n = 2 \cdot 2^n - 1$ so we need to determine particular solution z_n of (5) in two steps as a sum of two particular solutions z_{n1} and z_{n2}, first z_{n1} for the right side $2 \cdot 2^n$ and second z_{n2} for the right side -1. For the right side $2 \cdot 2^n$ we obtain particular solution

$$z_{n1} = \frac{2^{n+1}}{2 - r}$$

and for the right side -1 the solution

$$z_{n2} = \frac{1}{r - 1}.$$

Then the particular solution of (5) is sum of z_{n1} and z_{n2} i. e.

$$z_n = z_{n1} + z_{n2} = \frac{2^{n+1}}{2 - r} + \frac{1}{r - 1}$$

Finally we determine the particular solution which fulfils the additional following conditions: the first payment has value 1 and we obtain the following formula for the accumulated amount S_n for our immediate annuity

$$S_n = y_n + z_n = = \frac{r^{n+1}}{(r-2)(r-1)} + \frac{2^{n+1}}{2-r} + \frac{1}{r-1}\$\$$$

This procedure we can repeat for the more general case, i. e. for arbitrary real q and d. In the first step we then determine the formula for general expression for the n-th element of the annuity. We solve the difference equation

$$X_{n+1} - q \cdot X_n = d.$$

Copying preceding procedure step by step in more general form we first obtain the general solution y_n of the homogenous equation

$$y_n = cq^n.$$

But the right side is only $q_n = d$ so the particular solution is of the form $z_n = A$. Substituting $z_n = z_{n+1} = A$ into the non homogenous equation we become

$$A = \frac{d}{1-q}.$$

The required general solution is then the sum of y_n and z_n, i. e.

$$X_n = y_n + z_n = cq^n + \frac{d}{1-q}$$

and the particular solution which fulfils the additional following condition, the first payment has value 1, we can deduce from the condition

$$X_1 = 1 = cq^1 + \frac{d}{1-q}.$$

After some rearrangements solution for c is

$$c = \frac{1-q-d}{q(1-q)}$$

and finally formula for the particular solution which fulfils the additional following conditions, the first payment has value 1, is

$$X_n = \frac{1-q-d}{q(1-q)} q^n + \frac{d}{1-q}.$$

After some manipulations

$$X_n = \frac{(1-q-d)q^n + qd}{q(1-q)} = \frac{(1-q-d)q^{n-1} + d}{(1-q)} = \frac{d(1-q^{n-1}) + q^{n-1}(1-q)}{1-q} = \frac{d(1-q^{n-1})}{1-q} + q^{n-1}$$

And the final version of the formula is

$$X_n = \frac{d(q^{n-1}-1)}{q-1} + q^{n-1}.$$

Using the repetition of the preceding second step we can also find in this general case the formula for S_n, but the formula is too complicated, so we omit it.

In this note we showed the use of a method how to find the formula for accumulated amount for some special immediate annuities by solving the special type of non-homogenous linear difference equations of the first order.

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